DIFFERENTIAL GEOMETRY

DAILY SOLUTION

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1 *Lee - Smooth Manifolds - Problem(11-17)*

Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \mathbb{S}^1 \subset \mathbb{C}^n$ denote the n-torus. For each $j = 1, \ldots, n$, let $\gamma_j : [0,1] \to \mathbb{T}^n$ be the smooth curve segment

$$\gamma_j \ : \ t \ \mapsto \ \Big(1, \ \ldots, \ e^{2\pi i t}, \ \ldots, \ 1\Big),$$

such that $e^{2\pi i t}$ is in the jth place. Suppose $\omega \in \mathfrak{X}^*(\mathbb{T}^n)$ is a smooth covector field on $\mathbb{T}^n$ that is closed, meaning the component functions of ω satisfy

$$\frac{\partial \omega_i}{\partial x^j} \ = \ \frac{\partial \omega_j}{\partial x^i}$$

for each pair of indices i and j. Then ω is exact if and only if $\int_{\gamma_j} \omega$ vanishes for all j.

Proof If ω is an exact 1-form, then it is conservative (cf. Theorem 11-42), and therefore

$$\int_{\gamma_j} \omega \ = \ 0$$

for all j, because the curves γ_j are all closed. Conversely, suppose that $\int_{\gamma_j} \omega$ vanishes for all j. By Theorem(11-42), it is sufficient for us to prove ω is conservative. Then let $\gamma : [0,1] \to \mathbb{T}^n$ be a piecewise smooth closed curve with basepoint $\omega(0) = 1 = (1, \ldots, 1)$. We claim this choice is without loss of generality. Indeed, let $g : [0,1] \to \mathbb{T}^n$ be a path from $\gamma(0)$ to 1, and let $\widehat{\gamma}$ denote the path product $\overline{g} * \gamma * g$. Then $\widehat{\gamma}$ is also a closed curve, and by Proposition(11-34c) and parameter-independece of the line integral,

$$\int_{\widehat{\gamma}} \omega \ = \ \int_{\overline{g}} \omega + \int_{\gamma} \omega + \int_{g} \omega \ = \ \int_{\gamma} \omega,$$

with the second equality because $\int_{\overline{g}} \omega = -\int_g \omega$. Therefore ω vanishes over γ if and only if it vanishes over $\widehat{\gamma}$. Now consider the smooth universal covering map $\varepsilon : \mathbb{R}^n \to \mathbb{T}^n$ given by

$$\varepsilon \ : \ (x^1, \ldots, x^n) \ \mapsto \ \Big(e^{2\pi i x^1}, \cdots, e^{2\pi i x^n}\Big).$$

Fix a point $e \in \mathbb{R}^n$ in the fiber over $\gamma(0)$. Then by Lemma(1.1), there is a unique lift $\widetilde{\gamma} : [0,1] \to \mathbb{R}^n$ with $\widetilde{\gamma}(0) = e$, a piecewise smooth curve satisfying $\varepsilon \circ \widetilde{\gamma} = \gamma$, and by Proposition(11-34),

$$\int_{\gamma} \omega \ = \ \int_{\varepsilon \circ \widetilde{\gamma}} \omega \ = \ \int_{\widetilde{\gamma}} \varepsilon^* \omega.$$

Therefore to prove ω is conservative, we will show the integral of $\varepsilon^*\omega$ vanishes over the path $\widetilde{\gamma}$. The curve $\gamma = \varepsilon \circ \widetilde{\gamma}$ is closed, with $\varepsilon \circ \widetilde{\gamma}(0) = \varepsilon \circ \widetilde{\gamma}(1)$. The $\mathbb{Z}^n$-periodicity of ε implies $\widetilde{\gamma}(1) - \widetilde{\gamma}(0) \in \mathbb{Z}^n$, and hence

$$\widetilde{\gamma}(0), \widetilde{\gamma}(1) \ \in \ \mathbb{Z}^n. \tag{1}$$

because of our assumption on $\gamma(0)$. Now if ω is closed, then $\varepsilon^*\omega$ is closed: because ε is a local diffeomorphism (cf. Proposition 4-33), the pullback ε^* takes closed 1-forms to closed 1-forms (cf. Corollary 11-46). The Poincaré Lemma (cf. Theorem 11-49) implies $\varepsilon^*\omega$ is exact ($\varepsilon^*\omega$ is a closed 1-form on $\mathbb{R}^n$), and therefore $\varepsilon^*\omega = df$ for some smooth function $f \in C^\infty(\mathbb{R}^n)$. By the Fundamental Theorem of Line Integrals,

$$\int_{\widetilde{\gamma}} \varepsilon^* \omega \ = \ f \circ \widetilde{\gamma}(1) \ - \ f \circ \widetilde{\gamma}(0), \tag{2}$$

and in view of (2), what remains to be shown is that f is $\mathbb{Z}$-periodic in each coordinate when restricted to $\mathbb{Z}^n$. Now consider the curve γ_j defined above. From inspection of the covering ε, we can see

$$\widetilde{\gamma}_j \;:\; t \;\mapsto\; \left(\ell^1,\;\ldots,\;\ell^{j-1},\;\ell^j\;+\;t,\;\ell^{j+1},\;\ldots,\;\ell^n\right)$$

is a lift of γ_j for any choice of integers $\ell \in \mathbb{Z}^n$. Then equation (2) and our hypothesis on γ_j imply

$$\begin{aligned} 0 \;&=\; \int_{\gamma_j} \omega \;=\; \int_{\varepsilon \circ \widetilde{\gamma}_j} \omega \;=\; \int_{\widetilde{\gamma}_j} \varepsilon^* \omega \\ &=\; f \circ \widetilde{\gamma}_j(1) \;-\; f \circ \widetilde{\gamma}_j(0) \\ &=\; f(\ell^1,\;\ldots,\;\ell^j\;+\;1,\;\ldots,\;\ell^n) \;-\; f(\ell^1,\;\ldots,\;\ell^n), \end{aligned}$$

and with both j and ℓ arbitrary, this implies $f(k) = f(m)$ for all $k, m \in \mathbb{Z}^n$, as was to be shown. Therefore

$$\int_{\widetilde{\gamma}} \varepsilon^* \omega \;=\; f \circ \widetilde{\gamma}(1) \;-\; f \circ \widetilde{\gamma}(0) \;=\; 0,$$

and we conclude that ω is a conservative 1-form. ■

Lemma 1.1 *Suppose $\pi : E \to M$ is a smooth covering map. If $\gamma : [0,1] \to M$ is a piecewise smooth curve, then for any point $e_0 \in E$ such that $\pi(e_0) = \gamma(0)$, there exists a unique lift $\widetilde{\gamma} : [0,1] \to E$ such that $\widetilde{\gamma}(0) = e_0$, making the following diagram commute:*

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ I \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"'] & M. \end{tikzcd} \tag{3}$$

In other words, $\widetilde{\gamma}$ is a map satisfying $\gamma = \pi \circ \widetilde{\gamma}$, and since the path γ is piecewise smooth, $\widetilde{\gamma}$ is also piecewise smooth.

Proof For $i = 1, \ldots, k$, denote by $J_i = [a_{i-1}, a_i]$ the subintervals on which γ is smooth, such that $a_0 = 0$ and $a_k = 1$. By the Path Lifting Property of covering maps (cf. Proposition A-77), there is a unique continuous lift $\widetilde{\gamma}|_{J_1}$ of the smooth curve $\gamma|_{J_1}$, such that $\widetilde{\gamma}|_{J_1}(0) = e_0$ and $\pi \circ \widetilde{\gamma}|_{J_1} = \gamma|_{J_1}$. By hypothesis, the curve $\pi \circ \widetilde{\gamma}|_{J_1} = \gamma|_{J_1}$ is smooth, and because π is a local diffeomorphism (cf. Proposition 4-33) and $\widetilde{\gamma}|_{J_1}$ is continuous, Exercise(4-10) implies the lift $\widetilde{\gamma}|_{J_1}$ is smooth.

Now let $e_1 = \widetilde{\gamma}|_{J_1}(a_1)$ such that $\pi(e_1) = \gamma(a_1)$. Applying the Path Lifting Property again, this time for basepoint e_1, we get a unique lift $\widetilde{\gamma}|_{J_2}$ of the smooth curve $\gamma|_{J_2}$, such that $\widetilde{\gamma}|_{J_2}(a_1) = e_1$ and $\pi \circ \widetilde{\gamma}|_{J_2} = \gamma|_{J_2}$. By the same argument as before, this lift is smooth. Continuing on in this manner, we construct k smooth lifts $\widetilde{\gamma}|_{J_i}$, and these lifts combine to form the required lift $\widetilde{\gamma} : [0,1] \to E$ satisfying (3), and by construction, this lift is unique and satisfies $\widetilde{\gamma}(0) = e_0$. ■